Solution for HW 7 17-11-2016

§54) 3) Let $g(z)=\frac{1}{f(z)}$. Since $f(z)\neq 0\ \forall z\in R$, $g(z)$ is analytic on R. Since f and hence g are not constant on $\text{Int}(R)$, by max. principle, $|g(z)|$ attains its max. only on ∂R.
As a result, $|f(z)|=\frac{1}{|g(z)|}$ attains its min. only on ∂R.

6) Let $g(z)=e^{f(z)}$. Since $f(z)$ is analytic and non-constant on R, $g(z)$ is analytic and non-constant on R.
By Ex 3, $|g(z)|=|e^{f(z)}|=e^{u(x,y)}$ attains its min. only on ∂R.
Since $h(x)=e^x$ is increasing $\forall x\in\mathbb{R}$, $u(x,y)$ attains its min. only on ∂R.

7) $f(z)=e^z=e^x\cdot e^{iy}=e^x\cos y+ie^x\sin y$.
$\therefore\ u(x,y)=e^x\cos y$.
Since $x\in[0,1]$, $y\in[0,\pi]$, we have $1\le e^x\le e$ and $-1\le\cos y\le 1$. Hence we have $-e\le e^x\cos y\le e$.
The first equality holds iff $x=1, y=\pi$.
The second equality holds iff $x=1, y=0$.
So $f(z)$ attains its max. and min. only at the points $z=1$ and $z=1+\pi i$, which lie on ∂R.

§56) 2) Since $\text{Arg}(z)=\text{Arg}(re^{i\theta})=\theta$ is continuous $\forall\theta\in(-\pi,\pi)$,
$\lim_{n\to\infty}\text{Arg}(z_n)=\text{Arg}(\lim_{n\to\infty}z_n)=0$.
It is different from Example 2 in §55 since $\text{Arg}(z)$ is not continuous on end points.

4)
$$\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}$$
$$\Rightarrow\sum_{n=0}^{\infty}(re^{i\theta})^n=\frac{1}{1-re^{i\theta}}$$
$$\Rightarrow\sum_{n=0}^{\infty}r^ne^{in\theta}=\frac{1}{(1-r\cos\theta)-ir\sin\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}(r^n\cos n\theta + i r^n \sin n\theta) = \frac{1-r\cos\theta + i r\sin\theta}{(1-r\cos\theta)^2+(r\sin\theta)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} r^n\cos n\theta + i\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} + i\,\frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

By comparing the real and imaginary parts on both sides, we have

$$\sum_{n=0}^{\infty} r^n\cos n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} \quad \& \quad \sum_{n=0}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

$$\Rightarrow 1+\sum_{n=1}^{\infty} r^n\cos n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} \quad \& \quad 0+\sum_{n=1}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} r^n\cos n\theta = \frac{r\cos\theta-r^2}{1-2r\cos\theta+r^2} \quad \& \quad \sum_{n=1}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

6) $\sum_{n=1}^{\infty} z_n = S \Rightarrow \lim_{N\to\infty} |S_N - S| = 0$

$$\Rightarrow \lim_{N\to\infty} |\overline{S_N} - \overline{S}| = \lim_{N\to\infty} |\overline{S_N - S}|$$

$$= \lim_{N\to\infty} |S_N - S|$$

$$= 0.$$

Since $\overline{S_N} = \overline{\sum_{n=1}^{N} z_n} = \sum_{n=1}^{N} \overline{z_n}$, we have

$$\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$$